PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction
Balanced incomplete block designs, though have many optimal properties, do not fit well to many experimental situations as these designs require a large number of replications. Moreover, these designs are not available for all numbers of treatments and block sizes. To overcome these difficulties a class of binary, equireplicate and proper designs that are called Partially Balanced Incomplete Block (PBIB) designs were introduced. In these designs the variance of every estimated elementary contrast among treatment effects is not the same and hence the name PBIB designs. The definition of PBIB designs is based on the association scheme we, therefore, first give the concept of association scheme.

Association Scheme
Given \( v \) treatment symbols \( 1, 2, \ldots, v \), a relation satisfying the following conditions is called an \( m \)-class association scheme \((m \geq 2)\).

(i) Any two symbols are either 1st, 2nd, ..., or \( m \)th associates; the relation of association being symmetric, i.e., if the symbol \( \alpha \) is the \( i \)th associate of \( \beta \), then \( \beta \) is the \( i \)th associate of \( \alpha \).

(ii) Each symbol \( \alpha \) has \( n_i \) \( i \)th associates, the number \( n_i \) being independent of \( \alpha \),

(iii) If any two symbols \( \alpha \) and \( \beta \) are \( i \)th associates, then the number of symbols that are \( j \)th associates of \( \alpha \) and \( k \)th associate of \( \beta \) is \( p_{ijk} \) and is independent of the pair of \( i \)th associates \( \alpha \) and \( \beta \).

The numbers \( v, n_i \) and \( p_{ijk} \) \((i, j, k = 1, 2, \ldots, m)\) are called the parameters of the association scheme.

Example: Consider 12 treatment symbols denoted by numbers 1 to 12. Let us a form 3 group of 4 symbols each as follows: \((1,2,3,4), (5,6,7,8), (9,10,11,12)\). We now define (i) any two treatment symbols are first associates if they belong to the same group, (ii) any two treatment symbols are second associates if they belong to the different groups. Here \( v=12, n_1 = 3, n_2 = 8 \).

\[
\begin{align*}
p_{11}^1 &= 2, & p_{12}^1 &= 0, & p_{21}^1 &= 0, & p_{22}^1 &= 8, \\
p_{11}^2 &= 0, & p_{12}^2 &= 3, & p_{21}^2 &= 3, & p_{22}^2 &= 4.
\end{align*}
\]

It can be verified that these values of \( p_{jk}^i \) \((i, j, k = 1, 2)\) remain unchanged for any choice of two first or second associates. These parameters are usually written in the form of the following matrices.

\[
P^1 = \begin{pmatrix} p_{ij}^1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, \quad P^2 = \begin{pmatrix} p_{ij}^2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}.
\]

Given an association scheme for the \( v \) symbols, we now define a PBIB design as follows:
PBIB Design
Given an association scheme with m classes (m ≥ 2) we have a PBIB design with m associate
classes based on the association scheme, if the v treatment symbols can be arranged into b
blocks, such that
(i) Every symbol occurs at most once in a block.
(ii) Every symbol occurs in exactly r blocks.
(iii) If two symbols are \( i^{th} \) associates, then they occur together in \( \lambda_i \) blocks, the number \( \lambda_i \)
    being independent of the particular pair of \( i^{th} \) associates \( \alpha \) and \( \beta \).

The numbers \( v, b, r, k, \lambda_i \) \( (i=1,2,...,m) \) are called the parameters of the design. It can be easily
seen that
\[
\sum_{i=1}^{m} n_i \lambda_i = r(k - 1)
\]

Two-class association schemes and the two-associate PBIB designs have been extensively
studied in the literature and are simple to use. We, therefore, deal with some of these
schemes and the designs based on them in the following sections:

2. Some Two-class Association Schemes
Here we briefly define some of the well known two-associate class association schemes and
give their parameters.

(i) Group Divisible (GD) Association Scheme
Let \( v = mn \) symbols be arranged into \( m \) groups of \( n \) symbols each. A pair of symbols
belonging to the same group is first associates and a pair of symbols belonging to different
groups is second associates. This defines a GD association scheme and it has the following
parameters:
\[
n_1 = n-1, \quad n_2 = n(m-1) \]
\[
p^1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix}, \quad p^2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}
\]

A PBIB (2) design based on a GD scheme is called a GD design. A GD design is called
(i) singular if \( r\lambda_1 = 0 \);
(ii) semi-regular, if \( r\lambda_1 > 0 \), and \( rk-v\lambda_2 = 0 \);
(iii) regular, if \( r\lambda_1 > 0 \) and \( rk-v\lambda_2 > 0 \).

(ii) Triangular association scheme
Fill in an \( n \times n \) square array with \( v = n(n-1)/2 \) symbols in such a way that the positions in the
principal diagonal (running from upper left-hand to lower right-hand corner) are left blank,
the \( n(n-1)/2 \) positions above the principal diagonal are filled by so many symbols, and the
\( n(n-1)/2 \) positions below the principal diagonal are filled so that the array is symmetrical
about the principal diagonal. For any symbol \( \emptyset \), now, the first associates are the treatments
that occur in the same row (or in the same column) with \( \emptyset \) in the above arrangement. The
remaining symbols are defined as second associates of $\emptyset$. The triangular association scheme has the following parameters:

$$n_1 = 2(n-2), \quad n_2 = (n-2)(n-3)/2,$$

$$P^1 = \left[ \begin{array}{c} n-2 \\ n-3 \end{array} \right] \left[ \begin{array}{c} n-3 \\ n-3(n-4)/2 \end{array} \right], \quad P^2 = \left[ \begin{array}{c} 4 \\ 2n-8 \end{array} \right] \left[ \begin{array}{c} 2n-8 \\ (n-4)(n-5)/2 \end{array} \right] \quad (2.2)$$

(iii) Latin-Square, L$_i$ (i=2,3,...) - association scheme

Let $v = s^2$ symbols be arranged into an $s \times s$ square array and $i$-2 mutually orthogonal latin squares (MOLS) be super-imposed on the array. Two symbols are defined to be first associates if they occur in the same row, or column of the array or in position occupied by the same letter in any of the latin squares. This defines L$_i$ association scheme that has the following parameters:

$$n_1 = i(s-1), \quad n_2 = (s-i+1)(s-1),$$

$$P^1 = \left[ \begin{array}{c} (i-1)(i-2) + s-2 \\ (s-i+1)(i-1) \end{array} \right] \left[ \begin{array}{c} (s-i+1)(s-i) \\ (s-i+1)(s-i) \end{array} \right],$$

$$P^2 = \left[ \begin{array}{c} i(i-1) \\ i(s-i) \\ i(s-i) \\ (s-i)(s-i-1) + s-2 \end{array} \right] \quad (2.3)$$

3. Methods of Construction

It may be mentioned that as in the case of BIB designs, the complementary design of a PBIB with parameters $v, b, r, k, \lambda$ is also a PBIB design having the same association scheme with the parameters $v^* = v, b^* = b, r^* = b-r, k^* = v-k, \lambda_i^* = b-2r+\lambda_i$. We describe below some methods of constructing PBIB designs based on the above association schemes.

3.1 Group Divisible Designs

(1) Let $D$ be an incomplete block design. The design $D^*$ obtained from $D$ by interchanging the role of treatments and blocks is called the dual design of $D$. For example, if $D$ with parameters, $v = 4, b = 6, r = 3, k = 2$ is

<table>
<thead>
<tr>
<th>Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
</tr>
<tr>
<td>II</td>
</tr>
<tr>
<td>III</td>
</tr>
<tr>
<td>IV</td>
</tr>
<tr>
<td>V</td>
</tr>
<tr>
<td>VI</td>
</tr>
</tbody>
</table>

then its dual $D^*$ with parameters $v^* = 6, b^* = 4, r^* = 2, k^* = 3$ is

<table>
<thead>
<tr>
<th></th>
<th>I, II, III</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, II, III</td>
</tr>
<tr>
<td>2</td>
<td>I, IV, V</td>
</tr>
<tr>
<td>3</td>
<td>II, IV, VI</td>
</tr>
<tr>
<td>4</td>
<td>III, V, VI</td>
</tr>
</tbody>
</table>

Let $D$ be an affine resolvable BIB design with parameters $v, b, r, k, \lambda$. Obviously, then any two blocks of the same set have no common treatment while any two blocks from different sets have $k^2/v$ common treatments. Therefore, $D^*$, the dual design of $D$ is a GD design with the following parameters:

$$v^* = b, \quad b^* = v, \quad r^* = k, \quad k^* = r, \quad m = r, \quad n = b/r, \quad \lambda_1 = 0, \quad \lambda_2 = k^2/v,$$
where the first associates of the treatments of $D^*$ are the blocks belonging to the same group in $D$. The following series of BIB design are affine-resolvable:

(a) $v = s^2, b = s(s+1), r = s+1, k = s, \lambda = 1$;
(b) $v = s^3, b = s(s^2+s+1), r = s^2+s+1, k = s^2, \lambda = s+1; s$ being a prime or power of a prime in (a) and (b).

These yield the following series of GD designs:

(a) $v^* = s(s+1), b^* = s^2, r^* = s+1, m = s+1, n = s, \lambda_1 = 0, \lambda_2 = 1$;
(b) $v^* = s(s^2+s+1), b^* = s^3, r^* = s^2, k^* = s^2+s+1, m = s^2+s+1, n = s, \lambda_1 = 0, \lambda_2 = s$; $s$ being a prime or power of a prime in (a) and (b)'.

2. Let $D$ be a BIB design with parameters $v = m, b, r, k, \lambda$. Obtain a design $D^*$ from $D$ by replacing the $i$th treatment ($i = 1, 2, ..., v$) in $D$ by $n$ new treatment symbols $i_1, i_2, ..., i_n$. Evidently, $D^*$ is a group divisible design with the following parameters:

$\lambda_1 = r$, $\lambda_2 = \lambda$.

3. By omitting the blocks in which a particular treatment, say $\theta$, occurs from a BIB design with the parameters $v, b, r, k, \lambda = 1$, we obtain a GD design consisting of the remaining blocks with the parameters:

$\lambda_1 = r - 1, \lambda_2 = 1$.

In the $r$ blocks in which $\theta$ occurs, we find that on omitting $\theta$ they become disjoint and the remaining $v-1 = r(k-1)$ treatment symbols form $r$ groups each of $k-1$ symbols. This defines the GD association scheme on which the GD design is based.

3.2 Triangular Designs

(1) An obvious method of construction of a triangular design is to take the rows (or columns) of the association scheme as blocks of the design. Such a design will have the following parameters:

$v = n(n-1)/2, b = n, r = 2, k = n-1, \lambda_1 = 1, \lambda_2 = 0$.

It may be noted that this triangular design can also be obtained by dualising irreducible BIB design with parameters given below:

$\lambda_1 = n - 1, \lambda_2 = 2, \lambda' = 1$.

(2) If there exists a BIB design with the parameters

$v = (n-1)(n-2)/2, b = n(n-1)/2, r = n, k = n-2, \lambda = 2$,

then by dualizing it a triangular design with the parameters

$v^* = n(n-1)/2, b^* = (n-1)(n-2)/2, r^* = n-2, k^* = n, \lambda_1 = 1, \lambda_2 = 2$

can be constructed.
3.3 L₂ designs

3.3.1 Methods of Constructing L₂ designs

(1) Let \( v \) be a squared number, i.e. \( v = s^2 \). We write the \( s^2 \) treatment symbols in the following form:

\[
\begin{array}{cccccc}
1 & 2 & 3 & \ldots & s \\
&s+1&s+2&s+3&\ldots&2s \\
A &=& . & . & . & \ldots & . \\
. & . & . & . & . & \ldots & . \\
. & . & . & . & . & \ldots & . \\
(s-1)s+1&(s-1)s+2&(s-1)s+3&\ldots&s^2
\end{array}
\]

An incomplete block design with rows of \( A \) and columns of \( A \) as blocks is called a simple lattice design. A simple lattice has \( v = s^2 \), \( b = 2s \), \( r = 2 \), \( k = s \). It is easy to see that a simple lattice is an \( L_2 \) design with \( \lambda_1 = 1 \), \( \lambda_2 = 0 \).

(2) If \( s \) is a prime or a prime power, we can construct a series of \( L_2 \) designs as follows: we superimpose each latin square of the complete set of \( (s-1) \) mutually orthogonal latin squares on \( A \) defined above and form blocks with treatments which fall under the same letter of a latin square. This gives us an \( L_2 \) designs with parameters:

\[
v = s^2, \; b = s(s-1), \; r = s-1, \; k = s, \; \lambda_1 = 0, \; \lambda_2 = 1.
\]

3.3.2 Method of Constructing Lₙ (i>2) designs

A simple lattice is also called a square lattice. These designs have \( v = s^2 \). Let there exist \( i-2(i-2<s-1) \) mutually orthogonal latin squares of orders \( s \), and let us superimpose each of these squares on \( A \) as defined above in Sec. 3.3.1(1). Treating the rows of \( A \), columns of \( A \), symbols of \( A \) falling under same letter of 1st, 2nd, ..., \( (i-2) \)-th latin square as blocks, we get \( i \) blocks each of size \( s \). These \( i \) blocks constitute an \( L_i \) design having the parameters:

\[
v = s^2, \; b = s^i, \; r = s-1, \; k = s, \; \lambda_1 = 1, \; \lambda_2 = 0
\]

A large number of two-associate designs can be found in Bose, Clatworthy and Shrikhande (1954) and Clatworthy (1973).

4. Analysis of PBIB (2) designs

Let \( D \) be a PBIB design with two associate classes having the parameters

\[
v, \; b, \; r, \; k, \; \lambda_i, \; n, \; p_{jk}^1, \; \quad i,j,k = 1,2
\]

Consider the fixed effect model

\[
\text{Observation (y)} = \text{General mean (µ)} + \text{Treatment effect (τ)} + \text{Block effect (β)} + \text{Random error}
\]

where random errors are assumed to be identically independently normally distributed with mean zero and constant variance \( \sigma^2 \).

Minimization of the residual sum of squares with respect to the constants included in the model yields a set of normal equations which in view of the restrictions \( \Sigma \tau_i = 0 \) and \( \Sigma \beta_j = 0 \) can be solved to give
\[ \hat{\tau}_i = k \left[ B_2 Q_i - A_2 S_i(Q_i) \right] / (A_1 B_2 - A_2 B_1), \quad i = 1, 2, \ldots, v \]  \hspace{1cm} (4.3)

with
\[
\begin{align*}
A_1 &= r (k-1) + \lambda_2, \\
A_2 &= \lambda_2 - \lambda_1, \\
B_1 &= (\lambda_2 - \lambda_1) p_{i1}, \\
B_2 &= r (k-1) + \lambda_2 + (\lambda_2 - \lambda_1) (p_{i1} - p_{i1}^2),
\end{align*}
\]

and \( S_i(Q_i) \) is the sum of the adjusted totals of those treatments which are the first associates of treatment \( i \).

The adjusted treatment sum of squares is \( \sum_{i=1}^{v} \hat{\tau}_i Q_i = (SST_A, \text{say}) \)

\[ \text{Var} (\hat{\tau}_i - \hat{\tau}_m) = \frac{2k(A_2 + B_2)\sigma^2}{(A_1 B_2 - A_2 B_1)} = v_1, \text{say} \]  \hspace{1cm} (4.4)

if \( i \) and \( m \) are the first associates

\[ \frac{2kB_2 \sigma^2}{A_1 B_2 - A_2 B_1} = v_2, \text{say} \]  \hspace{1cm} (4.5)

if \( i \) and \( m \) are the second associates.

The average variance of all estimated elementary treatment contrasts is given by
\[ \text{A.V.} = \frac{n_1 v_1 + n_2 v_2}{n_1 + n_2} \]  \hspace{1cm} (4.6)

Details of the analysis are illustrated with the help of following example.

**EXAMPLE**

A varietal trial on wheat crop was conducted using a two-associate class P.B.I.B. design. The parameters of the design are \( v = b = 9, r = k = 3, \lambda_1 = 1, \lambda_2 = 0, n_1 = 6, n_2 = 2, \)
\[
p^1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad p^2 = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

The data along with the block contents are given below:

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Block contents/yields per plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(3) 59</td>
</tr>
<tr>
<td></td>
<td>(8) 56</td>
</tr>
<tr>
<td></td>
<td>(4) 53</td>
</tr>
<tr>
<td>II</td>
<td>(2) 35</td>
</tr>
<tr>
<td></td>
<td>(7) 33</td>
</tr>
<tr>
<td></td>
<td>(4) 40</td>
</tr>
<tr>
<td>III</td>
<td>(1) 48</td>
</tr>
<tr>
<td></td>
<td>(7) 42</td>
</tr>
<tr>
<td></td>
<td>(5) 42</td>
</tr>
<tr>
<td>IV</td>
<td>(7) 46</td>
</tr>
<tr>
<td></td>
<td>(8) 56</td>
</tr>
<tr>
<td></td>
<td>(9) 51</td>
</tr>
<tr>
<td>V</td>
<td>(4) 61</td>
</tr>
<tr>
<td></td>
<td>(5) 61</td>
</tr>
<tr>
<td></td>
<td>(6) 55</td>
</tr>
<tr>
<td>VI</td>
<td>(3) 52</td>
</tr>
<tr>
<td></td>
<td>(9) 53</td>
</tr>
<tr>
<td></td>
<td>(5) 48</td>
</tr>
<tr>
<td>VII</td>
<td>(1) 54</td>
</tr>
<tr>
<td></td>
<td>(8) 58</td>
</tr>
<tr>
<td></td>
<td>(6) 62</td>
</tr>
<tr>
<td>VIII</td>
<td>(2) 45</td>
</tr>
<tr>
<td></td>
<td>(9) 46</td>
</tr>
<tr>
<td></td>
<td>(6) 47</td>
</tr>
<tr>
<td>IX</td>
<td>(1) 31</td>
</tr>
<tr>
<td></td>
<td>(2) 27</td>
</tr>
<tr>
<td></td>
<td>(3) 35</td>
</tr>
</tbody>
</table>

Carry out the analysis.

**Analysis:** Compute

Grand Total \((G) = 59 + 56 + \ldots + 35 = 1296\)

No. of observations \((n) = 27\)
Grand Mean ($\bar{y}$) = $G/n = 1296/27 = 48$

No. of replications = 3
Block size (k) = 3
CF = $G^2/n = (1296)^2/27 = 62208$

<table>
<thead>
<tr>
<th>Treat/ Block no.</th>
<th>$T_i$</th>
<th>$B_j$</th>
<th>Block No’s in which treat $i$ occurs</th>
<th>$\sum B_j$</th>
<th>$\sum B_j/k$</th>
<th>$Q_i (2) - (6)$</th>
<th>$\hat{t}_i$</th>
<th>Adj. treat. mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>133</td>
<td>168</td>
<td>3,7,9</td>
<td>399</td>
<td>133</td>
<td>0</td>
<td>-5/18</td>
<td>47.72</td>
</tr>
<tr>
<td>2.</td>
<td>107</td>
<td>108</td>
<td>2,8,9</td>
<td>339</td>
<td>113</td>
<td>-6</td>
<td>-50/18</td>
<td>45.22</td>
</tr>
<tr>
<td>3.</td>
<td>146</td>
<td>132</td>
<td>1,6,9</td>
<td>414</td>
<td>138</td>
<td>8</td>
<td>73/18</td>
<td>52.06</td>
</tr>
<tr>
<td>4.</td>
<td>154</td>
<td>153</td>
<td>1,2,5</td>
<td>453</td>
<td>151</td>
<td>3</td>
<td>22/18</td>
<td>49.33</td>
</tr>
<tr>
<td>5.</td>
<td>151</td>
<td>177</td>
<td>3,5,6</td>
<td>462</td>
<td>154</td>
<td>-3</td>
<td>-23/18</td>
<td>49.28</td>
</tr>
<tr>
<td>6.</td>
<td>164</td>
<td>153</td>
<td>5,7,8</td>
<td>489</td>
<td>163</td>
<td>1</td>
<td>10/18</td>
<td>48.56</td>
</tr>
<tr>
<td>7.</td>
<td>121</td>
<td>174</td>
<td>2,3,4</td>
<td>393</td>
<td>131</td>
<td>-10</td>
<td>-89/18</td>
<td>43.06</td>
</tr>
<tr>
<td>8.</td>
<td>170</td>
<td>138</td>
<td>1,4,7</td>
<td>495</td>
<td>165</td>
<td>5</td>
<td>49/18</td>
<td>50.72</td>
</tr>
<tr>
<td>9.</td>
<td>150</td>
<td>93</td>
<td>4,6,8</td>
<td>444</td>
<td>148</td>
<td>2</td>
<td>13/18</td>
<td>48.72</td>
</tr>
</tbody>
</table>

Note: $T_1 = 48+54+31 = 133$, etc.; $B_1 = 59+53+56 = 168$, etc.
Total of Blocks, in which treatment 1 occurs, $\sum B_j = 132+174+93 = 399$, etc.

**Associates of Different Treatments**

1. First associate | (2,3,5,6,7,8)
   Second associate | (4,9)

2. First associate | (1,3,4,6,7,9)
   Second associate | (5,8)

3. First associate | (1,2,4,5,8,9)
   Second associate | (6,7)

4. First associate | (2,3,5,6,7,8)
   Second associate | (1,9)

5. First associate | (1,3,4,6,7,9)
   Second associate | (2,8)

6. First associate | (1,2,4,5,8,9)
(3,7) Second associate

(1,2,4,5,8,9) First associate

(3,6) Second associate

8

(1,3,4,6,7,9) First associate

(2,5) Second associate

9

(2,3,5,6,7,8) First associate

(1,4) Second associate

$$\hat{\tau}_i = \frac{k\{B_2Q_i - A_2S_1(Q_i)\}}{(A_1B_2 - A_2B_1)}$$ 

$$A_1 = \tau(k - 1) + \lambda_2 = 3(3-1) + 0 = 6$$

$$A_2 = \lambda_2 - \lambda_1 = 0 - 1 = -1$$

$$B_1 = (\lambda_2 - \lambda_1) p_{12}^2 = 0, \quad \text{as} \quad p_{12}^2 = 0$$

$$B_2 = r(k-1) + \lambda_2 + (\lambda_2 - \lambda_1) (p_{11}^2 - p_{11}^2)$$

$$= 3 \times 2 + 0 + (-1)(3-6)$$

$$= 6 + 3 = 9$$

Now $$A_1B_2 - A_2B_1 = 6 \times 9 - 0 = 54.$$  

Therefore, $$\hat{\tau}_i = 3 \left[ \frac{9Q_i + S_1(Q_i)}{54} \right]$$

$$= \frac{Q_i}{2} + \frac{S_1(Q_i)}{18}, \quad i = 1, 2, \ldots, 9$$

Now, $$S_1(Q_1) = \text{Sum of } Q_i's \text{ for those treatments which are first associates of treatment } 1,$$

$$= Q_2 + Q_3 + Q_5 + Q_6 + Q_7 + Q_8 = -6 + 8 - 3 - 10 + 1 + 5 = -5,$$

$$S_1(Q_2) = 4, \quad S_1(Q_3) = 1, \quad S_1(Q_4) = -5,$$

$$S_1(Q_5) = 4, \quad S_1(Q_6) = 1, \quad S_1(Q_7) = 1,$$

$$S_1(Q_8) = 4, \quad S_1(Q_9) = -5.$$  

Total S.S. (TSS) $$= \Sigma(\text{observation})^2 - \text{CF}$$

$$= 59^2 + 56^2 + \ldots + 35^2 - 62208$$

$$= 2490$$

Treatment S.S. unadjusted (SSTU) $$= (\Sigma T_i^2)/r - \text{CF}$$

$$= (133^2 + \ldots + 150^2)/3 - 62208$$

$$= 1094.67$$

Block S.S. unadjusted (SSBU) $$= (\Sigma B_j^2)/k - \text{CF}$$

$$= (168^2 + \ldots + 93^2)/3 - 62208$$

$$= 2268.$$
Treatment S.S. adjusted ($SST_A$) = $\sum \tau_i Q_i$

\[ = 0 \times (-5/18) + (-6) \times (-50/18) + ... + 2 \times (13/18) \]

\[ = 121.67. \]

Block S.S. adjusted ($SSB_A$) = $SST_A + SSB_U - SST_U$

\[ = 121.67 + 2268 - 109.67 \]

\[ = 1295. \]

Error S.S. ($SSE$) = $TSS - SSB_U - SST_A$

\[ = 2490 - 2268 - 121.67 \]

\[ = 100.33. \]

The analysis of variance Table is given below:

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>F</th>
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</thead>
<tbody>
<tr>
<td>Blocks (unadj.)</td>
<td>8</td>
<td>2268</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments (adj.)</td>
<td>8</td>
<td>121.67</td>
<td>15.2</td>
<td>1.52</td>
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<tr>
<td>Blocks (adj.)</td>
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<td>1295</td>
<td>161.87</td>
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<tr>
<td>Treatments (unadj)</td>
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<td></td>
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<tr>
<td>Error</td>
<td>10</td>
<td>100.33</td>
<td>10.033</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>26</strong></td>
<td><strong>2490</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table value of F (8,10) = 3.07 (at 5% level of significance)

Treatment effects are not significant.

SE(1) = Standard error of ($\hat{\tau}_i - \hat{\tau}_m$) = $\sqrt{[2 \times 3 \times (-1 + 9) \times MSE / 54]}$

\[ = \sqrt{(8/9) \times 10.033} \]

\[ = 2.99, \text{ if the treatments are first associates} \]

SE(2) = Standard error of ($\hat{\tau}_i - \hat{\tau}_m$) = $\sqrt{2 \times 3 \times 9 \times MSE / 54}$

\[ = \sqrt{10.033} \]

\[ = 3.17, \text{ if the treatments are second associates} \]

$CD_1 = t_{(0.05, 26)} \times SE(1) = 2.056 \times 2.99 = 6.14$, if two treatments are first associates

$CD_2 = t_{(0.05, 26)} \times SE(2) = 2.056 \times 3.17 = 6.51$, if two treatments are second associates

An estimate of average variance of elementary treatment contrast is

A.V. = $[6 \times 8.916 + 2 \times 10.030] / 8 = 9.195$

Average SE = 3.03

An unbiased estimate of the difference between two treatment effects ($\hat{\tau}_i - \hat{\tau}_m$) is:
A comparison of (\(\hat{\tau}_i - \hat{\tau}_m\)) with \(CD_1\) if two treatments are first associates and \(CD_2\) when treatments are second associates, indicates that treatments 2 and 3, 3 and 7, 4 and 7, 5 and 7, and 7 and 8 are significantly different.

For further details refer to Dey (1986). The same analysis can be carried out using SAS. The steps and analysis are given below:

```sas
data pbib;
input blk trt yld;
cards;
  1   3   59
  1   8   56
  1   4   53
  2   2   35
  2   7   33
  2   4   40
  3   1   48
  3   7   42
  3   5   42
  4   7   46
  4   8   56
  4   9   51
  5   4   61
  5   5   61
  5   6   55
  6   3   52
  6   9   53
  6   5   48
  7   1   54
  7   8   58
  7   6   62
  8   2   45
  8   9   46
  8   6   47
  9   1   31
  9   2   27
  9   3   35
```

<table>
<thead>
<tr>
<th>Treatment m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
<th>9</th>
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<td>1</td>
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<td>2.5</td>
<td>4.34</td>
<td>1.61</td>
<td>1.56</td>
<td>0.84</td>
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<td>4.11</td>
<td>4.06</td>
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<td>1.34</td>
<td>3.34</td>
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<table>
<thead>
<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
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<th>9</th>
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<td>5.5</td>
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<td>0.16</td>
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<tr>
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<td>9</td>
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<tr>
<td>8</td>
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<td>5.5</td>
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<td>5.66</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Bold figures indicate significant at 5% level.
proc print;
proc glm;
class blk trt;
model yld=blk trt/ss1 ss2;
LSMEANS trt/stderr pdiff;
run;

General Linear Models Procedure
Class Level Information

Class Levels Values
BLK 9 1 2 3 4 5 6 7 8 9
TRT 9 1 2 3 4 5 6 7 8 9

Number of observations in data set = 27
General Linear Models Procedure

Dependent Variable: YLD

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
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<tr>
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<td>149.35416667</td>
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<tr>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

R-Square  C.V.  Root MSE  YLD Mean
0.959705   6.599049 3.16754374 48.00000000

Source   DF   Type I SS   Mean Square   F Value   Pr > F
BLK      8    2268.00000000 283.50000000 28.26   0.0001
TRT      8    121.66666667 15.20833333 1.52    0.2641

Source   DF   Type II SS  Mean Square  F Value  Pr > F
BLK      8    1295.00000000 161.87500000 16.13  0.0001
TRT      8    121.66666667 15.20833333 1.52   0.2641

Least Squares Means

| TRT  | YLD     | Std Err | Pr > |T| | LSMEAN | LSMEAN | H0:LSMEAN=0 | Number |
|------|---------|---------|------|---|--------|--------|------------|--------|
| 1    | 47.7222222 | 2.1116958 | 0.0001 | 1 |
| 2    | 45.2222222 | 2.1116958 | 0.0001 | 2 |
| 3    | 52.0555556 | 2.1116958 | 0.0001 | 3 |
| 4    | 49.2222222 | 2.1116958 | 0.0001 | 4 |
| 5    | 46.7222222 | 2.1116958 | 0.0001 | 5 |
| 6    | 48.5555556 | 2.1116958 | 0.0001 | 6 |
| 7    | 43.0555556 | 2.1116958 | 0.0001 | 7 |
| 8    | 50.7222222 | 2.1116958 | 0.0001 | 8 |
| 9    | 48.7222222 | 2.1116958 | 0.0001 | 9 |
Pr > |T| H0: LSMEAN(i)=LSMEAN(j)

<table>
<thead>
<tr>
<th>i/j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
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<td>0.0280</td>
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<td>0.5182</td>
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<tr>
<td>9</td>
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<td>0.05182</td>
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</tr>
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</table>

NOTE: To ensure overall protection level, only probabilities associated with pre-planned comparisons should be used.

References